INVOLUTIONS AND GENERALIZED CENTROSYMMETRIC AND SKEW-CENTROSYMMETRIC MATRICES

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ABSTRACT. We present new results and simple short proofs for known theorems about generalized centrosymmetric matrices and generalized skew-centrosymmetric (centroskew) matrices. We also make a comparison between the identity matrix and a nontrivial involutory matrix K by studying the eigensystem of $K + M$, where M is a structured matrix, with the goal of expressing this system entirely in terms of the corresponding eigensystem of M. Also, properties of some parametric families of the form $K + \rho M$ are studied.

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1. INTRODUCTION

Let K be an $n \times n$ nontrivial (i.e., $K \neq \pm I$) involutory matrix, x an $n \times 1$ vector, and A an $n \times n$ matrix. A is called K-centrosymmetric (or K-symmetric) if $AK = KA$ and K-skew-centrosymmetric (or K-skew-symmetric) if $AK = -KA$; x is called K-symmetric if $Kx = x$ and K-skew-symmetric if $Kx = -x$. K-centrosymmetric and K-skew-centrosymmetric matrices were studied in [5, 6, 7].

Properties of the sum of two matrices have been studied by many researchers in many contexts with the objective of finding connections between the eigensystem of the summands and the eigensystem of the sum. In particular, the eigenvalues of the sum of the identity matrix I and another matrix is one of the first sums that one encounters in elementary linear algebra. A very useful relative of the identity is the counteridentity J, which is obtained from the identity by reversing the order of its columns. The identity and the counteridentity are both involutory matrices. The question we pose is: if K is a nontrivial involutory matrix, how closely is the eigensystem of the sum $K + M$ related to that of the matrix M? With no restrictions on M, it appears that little can be said about the connections between these eigensystems. However, we show that if M is K-centrosymmetric or K-skew-centrosymmetric, then the question has a simple explicit answer. One consequence of our analysis is the construction of an analytic homotopy $H(t)$, $0 \leq t \leq 1$, in the space of diagonalizable matrices, between $K = H(0)$ and any real skew-symmetric K-skew-centrosymmetric matrix $S = H(1)$ such that $H(t)$ has only real or pure imaginary eigenvalues for $0 \leq t \leq 1$. In some proofs we will depend on the fact [6] that eigenvectors of K-centrosymmetric matrices can be chosen to be K-symmetric or K-skew-symmetric, which implies if A is a K-centrosymmetric matrix with ω linearly independent eigenvectors, then ω linearly independent eigenvectors of A can be chosen to be K-symmetric or K-skew-symmetric. We present a simple short proof of this fact. We present simple short proofs of two propositions in

[5] when the matrices involved are over the field of real numbers or the field of complex numbers. We also present new results about K-centrosymmetric and K-skew-centrosymmetric matrices. In the special case when $K = J$, more results can be derived (see [1, 2, 3, 4]). We note that the results in this paper are generalizations of results about centrosymmetric and skew-centrosymmetric matrices we published in [1, 2, 3, 4].

Throughout this paper, we denote the identity matrix by I, the counteridentity matrix by J, an $n \times n$ nontrivial involutory matrix by K, the set of all $n \times 1$ vectors that are either K-symmetric or K-skew-symmetric by $\mathcal{E}$, and the transpose of a matrix A by A^T. Unless we say otherwise, all matrices in the paper will be $n \times n$. We denote $\sqrt{-1}$ by i. A pure imaginary number refers to a number of the form bi, where b is any real number (including zero). If x is an $n \times 1$ vector, then we let x^+ represent the K-symmetric part of x; i.e. $x^+ = \frac{1}{2}(x + Kx)$, and we let x^- represent the K-skew-symmetric part of x; i.e. $x^- = \frac{1}{2}(x - Kx)$. Note that $x = x^+ + x^-$. We denote the multiset of all eigenvalues of A by evals(A). If λ is an eigenvalue of A with a corresponding eigenvector x, we say (λ, x) is an eigenpair of A. As it is the case in [5], if C and D are multisets, we write $C = \pm D$ if the elements of C are the same as those of D up to sign, and we write $C = iD$ if $C = \{id \mid d \in D\}$. Such definitions respect multiplicity. If A is an $n \times n$ matrix, we let A_c represent the K-centrosymmetric part of A; i.e. $A_c = \frac{1}{2}(A + KAK)$, and we let A_{sc} represent the K-skew-centrosymmetric part of A; i.e. $A_{sc} = \frac{1}{2}(A - KAK)$. Note that $A = A_c + A_{sc}$. Finally, we note that the matrices we consider are over the field of complex numbers (or the field of real numbers).

2. K-skew-Centrosymmetric Matrices and K-Centrosymmetric Matrices

It is known [6] that if H is a K-centrosymmetric matrix with γ linearly independent eigenvectors, then γ linearly independent eigenvectors of H can be chosen to be K-symmetric or K-skew-symmetric. The following theorem presents a simple short proof of this fact.

Theorem 2.1. *Let H be a K-centrosymmetric matrix. If (λ, x) is an eigenpair of H, then either (λ, x^+) or (λ, x^-) is an eigenpair of H.*

Proof.

$$(2.1) \qquad Hx^+ + Hx^- = \lambda x^+ + \lambda x^-.$$

Multiplying by K yields

$$(2.2) \qquad Hx^+ - Hx^- = \lambda x^+ - \lambda x^-.$$

Adding Equations 2.1 and 2.2 yields

$$(2.3) \qquad Hx^+ = \lambda x^+.$$

Subtracting Equation 2.2 from Equation 2.1 yields

$$(2.4) \qquad Hx^- = \lambda x^-.$$

Note that the only vector that is both K-symmetric and K-skew-symmetric is the zero vector, and note also that x^+ and x^- cannot be both zero because $x = x^+ + x^-$ is an eigenvector. Thus, either x^+ or x^- is an eigenvector of H corresponding to λ. $\qquad\square$

Corollary 2.2. *If H is a K-centrosymmetric matrix with γ linearly independent eigenvectors, then γ linearly independent eigenvectors of H can be chosen from $\mathcal{E}$.*

Proof. Note that for every eigenvalue λ of H, we can choose an eigenvector that is either K-symmetric or K-skew-symmetric. Thus, γ eigenvectors of H can be chosen to be K-symmetric or K-skew-symmetric. $\qquad\square$

Theorem 2.3. *Let H be an $n \times n$ K-centrosymmetric matrix and let γ be the number of linearly independent eigenvectors of H. Then, H and KH share γ linearly independent eigenvectors, and λ is an eigenvalue of H if and only if λ or $-\lambda$ is an eigenvalue of KH.*

Proof. γ linearly independent eigenvectors of H can be chosen to be K-symmetric or K-skew-symmetric. Since KH is also K-centrosymmetric, then the same thing holds for KH. Now it is easy to prove that if z is K-symmetric, then (λ, z) is an eigenpair of H if and only if (λ, z) is an eigenpair of KH. Also, it is easy to prove that if z is skew-symmetric, then (λ, z) is an eigenpair of H if and only if $(-\lambda, z)$ is an eigenpair of KH. $\qquad\square$

Now we present simple short proofs for Propositions 3.1 and 4.1 of [5] when the matrices involved are over the field of real numbers or the field of complex numbers. When $K = J$, the results state the effect of reversing the rows/columns of centrosymmetric matrices and skew-centrosymmetric matrices on their eigenvalues. Note that unlike Proposition 3.1 of [5], our previous theorem mentions not only eigenvalues, but also eigenvectors.

Corollary 2.4. *Let H be a K-centrosymmetric matrix. Then $\operatorname{evals}(KH) = \pm \operatorname{evals}(H)$.*

The above theorem and corollary hold also for the case when KH is replaced by HK (note that $KH = HK$).

Theorem 2.5. *Let S be a K-skew-centrosymmetric matrix. Then $\operatorname{evals}(KS) = i \operatorname{evals}(S)$.*

Proof. Note first that $(KS)^2 = -S^2$, and KS is K-skew-centrosymmetric. Then recall that μ is an eigenvalue of A^2 if and only if $\pm\sqrt{\mu}$ is an eigenvalue of A. Also recall that the eigenvalues of K-skew-centrosymmetric matrices occur in pairs; i.e. if λ is an eigenvalue of such a matrix, then so is $-\lambda$. Moreover, λ and $-\lambda$ have the same multiplicity. It follows that λ is an eigenvalue of S of multiplicity m if and only if $i\lambda$ is an eigenvalue of KS of multiplicity m. $\qquad\square$

The above theorem holds also for the case when KS is replaced by SK (note that $KS = -SK$). We note also that the converse of the previous theorem and the converse of the previous corollary hold in the case when H, S, and K are Hermitian. A nice proof of that can be found in [7].

Theorem 2.6. *Let M be an $n \times n$ matrix and let (λ, x) be an eigenpair of M. Then*
(a) $(\lambda, x^+ - x^-)$ is an eigenpair of $M_c - M_{sc} = KMK$.

(b) If x is K-symmetric, then (λ, x^+) is an eigenpair of M_c and $(0, x^+)$ is an eigenpair of M_{sc}.
(c) If x is K-skew-symmetric, then (λ, x^-) is an eigenpair of M_c and $(0, x^-)$ is an eigenpair of M_{sc}.
(d) If M is K-skew-centrosymmetric and nonsingular, and x is not K-symmetric, then (λ^2, x^-) is an eigenpair of M^2.
(e) If M is K-skew-centrosymmetric and nonsingular, and x is not K-skew-symmetric, then (λ^2, x^+) is an eigenpair of M^2.

Proof. $Mx = \lambda x$ if and only if

$$M_c x^+ + M_c x^- + M_{sc} x^+ + M_{sc} x^- = \lambda x^+ + \lambda x^-$$

and (after multiplying by K)

$$M_c x^+ - M_c x^- - M_{sc} x^+ + M_{sc} x^- = \lambda x^+ - \lambda x^-.$$

Part (a) follows from the second equation. Note that $x^+ - x^- \neq 0$, because $x \neq 0$.
Now if x is K-symmetric, then $x^- = 0$ and the first two equations become

$$M_c x^+ + M_{sc} x^+ = \lambda x^+,$$

$$M_c x^+ - M_{sc} x^+ = \lambda x^+.$$

Adding and subtracting the above two equations yields Part (b).
Now if x is K-skew-symmetric, then $x^+ = 0$ and the first two equations become

$$M_c x^- + M_{sc} x^- = \lambda x^-,$$

$$-M_c x^- + M_{sc} x^- = -\lambda x^-.$$

Adding and subtracting the above two equations yields Part (c).
Now if M is K-skew-centrosymmetric , then $M_c = 0$ and the first two equations become

$$M_{sc} x^+ + M_{sc} x^- = \lambda x^+ + \lambda x^-,$$

$$-M_{sc} x^+ + M_{sc} x^- = \lambda x^+ - \lambda x^-.$$

Adding and subtracting the above two equations yields

$$M_{sc} x^- = \lambda x^+,$$

$$M_{sc} x^+ = \lambda x^-.$$

Thus,

$$M_{sc} M_{sc} x^- = \lambda M_{sc} x^+ = \lambda^2 x^-.$$

$\square$

Note that the case when M is K-centrosymmetric was handled in Theorem 2.1.
It is known that if $\lambda \neq 0$ is an eigenvalue of a K-skew-centrosymmetric matrix, then λ cannot have a K-symmetric or a K-skew-symmetric eigenvector. But, if the matrix is also real skew-symmetric and K is real, then we have the following theorem.

Theorem 2.7. *Let K be an $n \times n$ real involutory matrix, S an $n \times n$ real skew-symmetric K-skew-centrosymmetric matrix, and $(\lambda \neq 0, x + iy)$ an eigenpair of S, where x and y are real n-vectors. Then, x is K-symmetric (respectively K-skew-symmetric) if and only if y is K-skew-symmetric (respectively K-symmetric).*

Proof. Let $(\lambda = bi, z = x + iy)$ be an eigenpair of S, where x and y are real, and assume $b \neq 0$. Then

$$(2.5) \qquad Sx + iSy = -by + ibx.$$

This implies

$$(2.6) \qquad KSx + iKSy = ibKx - bKy.$$

Thus,

$$(2.7) \qquad -SKx - iSKy = ibKx - bKy.$$

Now if x is K-symmetric, then

$$(2.8) \qquad -Sx - iSKy = ibx - bKy.$$

Now add Equations (2.5) and (2.8), to get

$$(2.9) \qquad i(Sy - SKy) = 2ibx - b(y + Ky).$$

Thus, $-b(y + Ky) = 0$. Since $b \neq 0$, it follows that $Ky = -y$.
  Now if y is K-skew-symmetric, then from (2.7), we get

$$(2.10) \qquad -SKx + iSy = ibKx + by.$$

Now subtract Equation (2.10) from Equation (2.5), to get

$$(2.11) \qquad S(x + Kx) = -2by + ib(x - Kx).$$

Thus, $b(x - Kx) = 0$. Since $b \neq 0$, it follows that $Kx = x$. The rest of the proof is similar. $\qquad \square$

In Section 6 of [2], we found useful and simple orthogonal transformations between centrosymmetric matrices and skew-centrosymmetric matrices of even order. Using such transformations, we can transform every skew-centrosymmetric singular value or determinant problem of even order to a centrosymmetric singular value or determinant problem of even order and vice versa. Moreover, we can transform every linear system in which the matrix of coefficients is centrosymmetric of even order to a linear system in which the matrix of coefficients is skew-centrosymmetric of even order, and vice versa. We leave it as an open question to find similar orthogonal transformations between K-centrosymmetric matrices and K-skew-centrosymmetric matrices of even order. We also leave it for future work to generalize the results in [4] about rank-one perturbations of centrosymmetric matrices to K-centrosymmetric matrices.

3. K-Centrosymmetric and K-Skew-Centrosymmetric Summands

In this section we analyze the relationship between the eigenvalues and eigenvectors of $K + S$ and the eigenvalues and eigenvectors of S, where S is either K-centrosymmetric or K-skew-centrosymmetric. The proof of the following theorem is straightforward, and hence omitted.

Theorem 3.1. *Let S be an $n \times n$ nonzero K-centrosymmetric matrix, let γ be the number of linearly independent eigenvectors of S, and let $A = K + S$. Then*

(a) A and S share γ linearly independent eigenvectors that belong to $\mathcal{E}$.

(b) If x is K-symmetric, then (λ, x) is an eigenpair of S if and only if $(1 + \lambda, x)$ is an eigenpair of A.

(c) If x is K-skew-symmetric, then (λ, x) is an eigenpair of S if and only if $(-1 + \lambda, x)$ is an eigenpair of A.

Observe that the ± 1's in the preceding proposition are the eigenvalues of the matrix K. Thus, if S has n linearly independent eigenvectors, then the eigenvalues of K can be ordered as $\mu_1, \mu_2, \ldots, \mu_n$ in such a way that the eigenvalues of $K + S$ are exactly $\mu_j + \lambda_j$, $j = 1, \ldots, n$, where the λ_j's are the eigenvalues of S. This is the best relationship between the eigenvalues of the summands and the eigenvalues of the sum that we can hope for.

Now we handle a special case of K-centrosymmetric matrices.

Theorem 3.2. *Let ρ be a nonzero real number, u an $n \times 1$ K-symmetric real vector such that $\|u\|_2 = 1$, K a nontrivial symmetric involution, and $H_\rho = K + \rho u u^T$. Then*

(a) *The eigenvectors of $u u^T$ are eigenvectors of $H_\rho{}^2$, and $u u^T$ and H_ρ share n linearly independent eigenvectors.*

(b) *If λ is an eigenvalue of H_ρ, then $\lambda = 1 + \rho$ or $\lambda = \pm 1$, where the ± 1 are the eigenvalues of K.*

(c) $\det(H_\rho) = \pm(1 + \rho)$.

(d) *If $\rho \neq -1$, then H_ρ is nonsingular and $H_\rho^{-1} = H_\sigma$, where $\sigma = \frac{-\rho}{1+\rho}$.*

Proof. First, note that the matrix $\rho u u^T$ has two distinct eigenvalues 0 (of multiplicity $n - 1$) and ρ. Now since the matrix $\rho u u^T$ is K-centrosymmetric, then Theorem 3.1 applies. You may also depend on the fact that $H_\rho^2 = I + (2\rho + \rho^2) u u^T$. For more details, we refer the reader to the proof of Theorem 4.1 in [1]. $\qquad \square$

The situation with a K-skew-centrosymmetric summand S is not so clear as in the K-centrosymmetric case. Nonetheless, the eigensystem of $K + S$ can be largely determined in terms of S.

Theorem 3.3. *Let S be an $n \times n$ nonzero K-skew-centrosymmetric matrix and let $A = K + S$. Then*

(a) *Every eigenvector of S is an eigenvector of A^2.*

(b) μ *is an eigenvalue of S if and only if $\pm\sqrt{\mu^2 + 1}$ is an eigenvalue of A.*

(c) *If, in addition, S is also skew-symmetric, then the eigenvalues of A are either real or pure imaginary.*

Proof.
$$A^2 = (S + K)(S + K) = S^2 + SK + KS + K^2 = S^2 + SK - SK + I = S^2 + I$$
from which (a) follows. Furthermore, if λ is an eigenvalue of A, then $\lambda^2 = \mu^2 + 1$, for some eigenvalue μ of S. Thus, (b) is proved. Now if S is also skew-symmetric and if λ is an eigenvalue of A, then $\lambda^2 = 1 + \mu^2$, for some eigenvalue $\mu = bi$ of S, where $b \in \mathbb{R}$. Thus, $\lambda^2 = 1 - b^2$ and hence, $\lambda = \pm\sqrt{1 - b^2}$. Therefore, λ is real if and only if $|b| \leq 1$ and pure imaginary if and only if $|b| > 1$. $\qquad \square$

Now let S be a real skew-symmetric and K-skew-centrosymmetric and let $A = K + S$. Since S is diagonalizable, then so is A^2. It follows that A^2 has no Jordan blocks of order greater than 1, so the same is true for A. Therefore, A is diagonalizable. Note also that the eigenvalues of A are real or pure imaginary, which enables us to make an interesting homotopy construction.

Example 3.4. We construct an analytic homotopy $H(t)$, $0 \leq t \leq 1$, in the (topological) space of diagonalizable matrices, between $K = H(0)$ and any $n \times n$ real skew-symmetric and K-skew-centrosymmetric $S = H(1)$ such that $H(t)$ has only real or pure imaginary eigenvalues for $0 \leq t \leq 1$. Specifically, define

$$H(t) = (1-t)K + tS.$$

Clearly $H(0) = K$ and $H(1) = S$ and $H(t)$ is analytic in t. Furthermore, for $0 < t < 1$, we have

$$H(t) = (1-t)\left(K + \frac{t}{(1-t)}S\right).$$

Now $\frac{1}{1-t}H(t)$ is diagonalizable (which implies $H(t)$ is diagonalizable) and by Theorem 3.3 its eigenvalues are real or pure imaginary.

Homotopies have applications in both linear and nonlinear systems of equations. The most significant application to linear systems goes back to the early 1990's, when Liu and others including Golub, worked on methods for finding eigenvalues of matrices. The idea is that with a homotopy of the form $M(\lambda) = \lambda A + (1-\lambda)B$, one can start with a simple matrix, e.g., diagonal B at $\lambda = 0$ and use the fact that the eigenvalues of M are continuous functions of λ. Since those at $\lambda = 0$ are known, one attempts to compute the eigenvalues of $M(\lambda)$ by *continuation methods* such as Ode Solvers as λ goes from 0 to 1 and in the end find the eigenvalues of the harder matrix A. (Moreover, these methods are important because they are naturally implemented on parallel computers.)

The relationship between the eigenvalues of $K + D$ and the eigenvalues of D, where $K = J$ and D is a matrix with zeros everywhere except possibly on the main diagonal or the main counterdiagonal (the positions which proceed diagonally from the last entry in the first row to the first entry in the last row) can be found in Section 3 of [1]. For the case when K is not a multiple of I or J, the relationship between the eigen structure of $K + D$ and the eigen structure of D remains an open question that we propose for our readers to investigate.

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