CENTROSYMMETRIC AND SKEW–CENTROSYMMETRIC MATRICES AND REGULAR MAGIC SQUARES

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Abstract. We prove results about the eigen structure and the singular values of centrosymmetric, skew–centrosymmetric, and doubly skew matrices, and about regular magic squares.

1. Introduction

Goldstein [7] reduced the eigen problem of a Hermitian persymmetric matrix to an eigen problem of a real symmetric matrix of the same size. We generalize his theorem to a wider class of matrices. The proof of our theorem is simple and valid for both even and odd orders. We present a linear orthogonal transformation between centrosymmetric matrices and skew–centrosymmetric matrices of even order. This transformation enables us to apply facts about centrosymmetric matrices (e.g. symmetric Toeplitz matrices), which are extensively studied, to skew–centrosymmetric matrices (e.g. skew–symmetric Toeplitz matrices), and vice versa. It is known that if H is a centrosymmetric matrix with γ linearly independent eigenvectors, then γ linearly independent eigenvectors of H can be chosen to be symmetric or skew-centrosymmetric. We present a simple short proof of this fact. We also present several facts about the eigen structure and the singular values of centrosymmetric matrices and skew–centrosymmetric matrices. Newman [11] is interested in the eigen structure of regular magic squares. He made some observations on their eigen structure. Mattingly [10] proved regular magic squares of even order are singular. We present a new proof of that, and also we present new properties of regular magic squares.

We employ the following notation. We denote the transpose of a matrix A by A^T and the Hermitian transpose by A^*. We use evals(A) to denote the eigenvalues of A (with multiplicities). We denote $\sqrt{-1}$ by i. Throughout this paper, we let $\delta = \frac{n}{2}$ and $\zeta = \frac{n-1}{2}$, and we let $\mathcal{E}$ be the set of all $n \times 1$ vectors that are either symmetric or skew–symmetric. If x is an $n \times 1$ vector, then we let x^+ represent the symmetric part of x; i.e. $x^+ = \frac{1}{2}(x + Jx)$, where J is the $n \times n$ counteridentity matrix, and we let x^- represent the skew–symmetric part of x; i.e. $x^- = \frac{1}{2}(x - Jx)$. Throughout this paper, we denote the identity matrix by I and the counteridentity matrix by J. We refer the reader to [1] for the definitions used in this paper without being introduced.

There are various kinds of symmetries that we will use in this paper. For convenience, we summarize them in the following definition.

Definition 1.1. Let A be an $n \times n$ matrix. Then, A is *persymmetric* if $JAJ = A^T$, *doubly symmetric* if it is symmetric and persymmetric, *centrosymmetric* if $JAJ = A$, *skew–centrosymmetric* if $JAJ = -A$, and *doubly skew* if it is skew–symmetric and skew–centrosymmetric.

Persymmetric matrices have applications in many fields including communication theory, statistics, physics, harmonic differential quadrature, differential equations, numerical analysis, engineering, sinc methods, magic squares, and pattern recognition. For applications of these matrices, see [**12, 6, 5, 10, 8, 4**]. Doubly skew matrices have applications in harmonic differential quadrature, sinc methods, and other fields. We note that many results for centrosymmetric and skew–centrosymmetric matrices have been generalized to wider classes of matrices that arise in a number of applications.

Definition 1.2. A *magic square* of order n is an $n \times n$ matrix whose elements are the integers 1 through n^2 and such that all rows and columns, the main diagonal, and the main counterdiagonal, have the same sum $\mu = \frac{n^3+n}{2}$. A magic square $A = (a_{ij})$ of order n is called *regular* if it satisfies $a_{i,j} + a_{n-i+1,n-j+1} = n^2 + 1$, $i = 1, \cdots, n; j = 1, \cdots, n$.

Lemma 1.3. *Let S be an $n \times n$ skew–centrosymmetric matrix. If n is even, then S can be written as*

$$S = \begin{bmatrix} A & -JCJ \\ C & -JAJ \end{bmatrix},$$

where A, J and C are $\delta \times \delta$. If, in addition, S is skew–symmetric, then A is skew–symmetric and C is persymmetric. If n is odd, then S can be written as

$$S = \begin{bmatrix} A & x & -JCJ \\ y^T & 0 & -y^T J \\ C & -Jx & -JAJ \end{bmatrix},$$

where A, J, and C are $\zeta \times \zeta$, and x and y are $\zeta \times 1$. If, in addition, S is skew–symmetric, then $y = -x$, A is skew–symmetric, and C is persymmetric.

A convenient summary of some classical results for centrosymmetric matrices (mostly special cases of results proved in [**3**]) is given in Theorem 2.3 of [**1**], which uses the notation

$$H = \begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix} \qquad \text{or} \qquad H = \begin{bmatrix} A & x & JCJ \\ y^T & q & y^T J \\ C & Jx & JAJ \end{bmatrix} \qquad (1.1)$$

for the usual partitioned form of a centrosymmetric matrix H, depending on whether n is even or odd. Clearly if H is also symmetric then A is symmetric and C is persymmetric, and in the odd case $x = y$ also.

The following fact can be found in [**9**].

Proposition 1.4. *Let S be a skew–centrosymmetric matrix. If (λ, x) is an eigenpair of S, then $(-\lambda, Jx)$ is an eigenpair of S. Moreover, λ and $-\lambda$ have the same multiplicity.*

The following theorem was proved in [**10**].

Theorem 1.5. *Let A be a regular magic square of order n, let e be the $n \times 1$ constant vector of ones, and let $\mu = \frac{n^3+n}{2}$. Then*

(1) A can be written as $A = Z + \frac{\mu}{n}ee^T$, where Z is skew–centrosymmetric.

(2) (μ, e) is an eigenpair of A and $(0, e)$ is an eigenpair of Z. Moreover, μ is the largest eigenvalue of A in magnitude and it is simple.

(3) All eigenpairs of A are the same as those of Z except that $(0, e)$ is replaced by (μ, e).

2. Reduction of Complex Eigen Problems

Theorem 2.1. *Let H be a real centrosymmetric matrix and S a real skew–centrosymmetric matrix, and assume all eigenvalues of $H + iS$ are real. Then the eigenvalues of $H \pm iS$ are the same as the eigenvalues of $H + JS$ and the same as the eigenvalues of $H - JS$. Moreover, if (λ, e) is an eigenpair of $H + JS$, then $(\lambda, e + iJe)$ is an eigenpair of $H + iS$, and if (μ, f) is an eigenpair of $H - JS$, then $(\mu, f - iJf)$ is an eigenpair of $H + iS$.*

Proof. The eigenvalues of $H + JS$ are the same as those of $H - JS$, since $\det(H + JS - \lambda I) = \det\big(J(H + JS - \lambda I)J\big) = \det(H - JS - \lambda I)$. Now, $(H + iS)(x + iy) = \lambda(x + iy)$ if and only if

$$\begin{bmatrix} H & -S \\ S & H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

Theorem 2.3 of [1] applies. Therefore, the eigenvalues of $H + iS$ are the eigenvalues of $H + JS$ and the eigenvalues of $H - JS$. But, the eigenvalues of $H + JS$ are the same as the eigenvalues of $H - JS$. Hence, λ is an eigenvalue of $H + iS$ if and only if λ is an eigenvalue of $H + JS$ if and only if λ is an eigenvalue of $H - JS$. Moreover, the eigenvectors of $H + iS$ can be obtained from the eigenvectors of $H + JS$ or from the eigenvectors of $H - JS$. $\square$

Thus, we can transform the complex eigen problem $(H \pm iS)z = \lambda z$ to the real eigen problem $(H + JS)w = \lambda w$. Moreover, if λ is an eigenvalue of $H \pm iS$, then we can choose an eigenvector $z = x + iy$ of λ such that $y = Jx$ or $y = -Jx$, where x and y are real.

Corollary 2.2. *Let H be a real centrosymmetric matrix and S a real skew–centrosymmetric matrix, and assume all eigenvalues of $H + iS$ are real. Then*

$$\det(H + iS) = \det(H - iS) = \det(H + JS) = \det(H - JS).$$

Corollary 2.3. *Let H be a real doubly symmetric matrix and S a real doubly skew matrix. Then the four matrices $H \pm iS$ and $H \pm JS$ all have the same eigenvalues, and hence, since they are all normal, they also have the same singular values.*

Remarks. Goldstein's (see [7]) reduction procedure follows directly from the above results. We note that a different generalization of the results of [7] is given in [13].

Corollary 2.4. *Let S be a real doubly skew matrix. Then, $\mathrm{evals}(JS) = i \cdot \mathrm{evals}(S)$. Moreover, if λ is an eigenvalue of S, then we can choose an eigenvector $z = x + iy$ of λ such that $y = Jx$ or $y = -Jx$.*

Corollary 2.5. *Let S be a real skew-centrosymmetric matrix and H a real centrosymmetric matrix, and assume all eigenvalues of $S + iH$ are pure imaginary. Then λ is an eigenvalue of $S + iH$ if and only if $-i\lambda$ is an eigenvalue of $H + JS$.*

3. Properties of Centrosymmetric/Skew–centrosymmetric Matrices

3.1. Eigenvalues and eigenvectors.

Proposition 3.1. *The transformation L defined by $L(M) = JM$ is a bijection on skew–centrosymmetric matrices and a bijection between doubly skew matrices and symmetric skew–centrosymmetric matrices.*

Theorem 3.2. *Every symmetric skew–centrosymmetric eigenvalue/determinant/ inverse problem is equivalent to a corresponding doubly skew problem and vice versa.*

Proof. If S is doubly skew (see Corollary 2.4), then $\mathrm{evals}(JS) = i \cdot \mathrm{evals}(S)$. Now, let T be symmetric skew–centrosymmetric. Then, JT is doubly skew. Thus, $\mathrm{evals}(T) = \mathrm{evals}\big(J(JT)\big) = i \cdot \mathrm{evals}(JT)$. Now let S be doubly skew. Then JS is symmetric skew–centrosymmetric. Note that $\mathrm{evals}(S) = -i \cdot \mathrm{evals}(JS)$. Now let T be a nonsingular symmetric skew–centrosymmetric matrix and let $H = JT$. Note that H is doubly skew and $T^{-1} = H^{-1}J$ and $\det(T) = \det(H)$ if $n \mod 4 = 0$ or 1, and $\det(T) = -\det(H)$ if $n \mod 4 = 2$ or 3. The rest of the proof is similar. $\square$

The following proposition is useful if we are interested only in the magnitude of the eigenvalues.

Proposition 3.3. *If S is doubly skew or skew-symmetric centrosymmetric or symmetric skew–centrosymmetric, then S^2 is symmetric centrosymmetric.*

Theorem 3.4. *Let n be even, let $E = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix} \right]$, where I is $\delta \times \delta$, and let H be an $n \times n$ matrix. Then EH is skew–centrosymmetric (respectively centrosymmetric) if and only if H is centrosymmetric (respectively skew–centrosymmetric).*

Thus, several centrosymmetric (respectively skew–centrosymmetric) problems (such as determinant/inverse) can be transformed to corresponding skew–centrosymmetric (respectively centrosymmetric) problems.

It is known that if H is a centrosymmetric matrix with γ linearly independent eigenvectors, then γ linearly independent eigenvectors of H can be chosen to be symmetric or skew–centrosymmetric. The following theorem presents a simple short proof of this fact.

Theorem 3.5. *Let H be a centrosymmetric matrix. If (λ, x) is an eigenpair of H, then either (λ, x^+) or (λ, x^-) is an eigenpair of H.*

Proof. $Hx^+ + Hx^- = \lambda x^+ + \lambda x^-$. Thus, $Hx^+ - Hx^- = \lambda x^+ - \lambda x^-$. Hence, $Hx^+ = \lambda x^+$ and $Hx^- = \lambda x^-$. $\square$

Corollary 3.6. *If H is a centrosymmetric matrix with γ linearly independent eigenvectors, then γ linearly independent eigenvectors of H can be chosen from $\mathcal{E}$.*

Corollary 3.7. *Let S be a nonsingular skew–centrosymmetric matrix. If (λ, x) is an eigenpair of S, then either (λ^2, x^+) or (λ^2, x^-) is an eigenpair of S^2.*

It is known that if $\lambda \neq 0$ is an eigenvalue of a skew–centrosymmetric matrix, then λ can not have a symmetric or a skew–symmetric eigenvector. But, if the matrix is also skew–symmetric, then we have the following theorem.

Theorem 3.8. *Let S be an $n \times n$ real doubly skew matrix. If $(\lambda \neq 0, x + iy)$ is an eigenpair of S, where x and y are real, then x is symmetric (respectively skew–symmetric) if and only if y is skew–symmetric (respectively symmetric).*

Proof. Let $(\lambda = bi, z = x + iy)$ be an eigenpair of S, where x and y are real, and assume $b \neq 0$. Then

$$Sx + iSy \;=\; -by + ibx. \tag{3.1}$$

This implies

$$JSx + iJSy \;=\; ibJx - bJy. \tag{3.2}$$

Thus,

$$-SJx - iSJy \;=\; ibJx - bJy. \tag{3.3}$$

Now if x is symmetric, then

$$-Sx - iSJy \;=\; ibx - bJy. \tag{3.4}$$

Now add (3.1) and (3.4), to get

$$i(Sy - SJy) \;=\; i2bx - b(y + Jy). \tag{3.5}$$

Thus, $-b(y + Jy) = 0$. Since $b \neq 0$, it follows that $Jy = -y$.
   Now if y is skew–symmetric, then from (3.3), we get

$$-SJx + iSy \;=\; ibJx + by. \tag{3.6}$$

Now subtract (3.6) from (3.1), to get

$$S(x + Jx) \;=\; -2by + ib(x - Jx). \tag{3.7}$$

Thus, $b(x - Jx) = 0$. Since $b \neq 0$, it follows that $Jx = x$. The rest of the proof is similar. $\qquad\square$

   The following theorem shows that reversing the rows/columns of a skew–centrosymmetric matrix results in multiplying the eigenvalues by i.

Theorem 3.9. *Let S be a skew–centrosymmetric matrix. Then* $\mathrm{evals}(JS) = i \cdot \mathrm{evals}(S)$.

Proof. The proof follows from Proposition 1.4 and the fact that $(JS)^2 = -S^2$. $\quad\square$

   More properties of (skew–)centrosymmetric matrices can be found in [**2**].

3.2. Singular Values.
   The following result is a special case of a result proved in [**3**], but the statement given there refers to eigenvalues of H^*H rather than singular values of H.

Theorem 3.10. *Let H be an $n \times n$ centrosymmetric matrix and let H be written as in Equation (1.1). Then*

(i) *If n is even, then the singular values of H are the nonnegative square roots of the eigenvalues of M_1 and the nonnegative square roots of the eigenvalues of M_2, where*

$$M_1 = (A - JC)^*(A - JC) \qquad and \qquad M_2 = (A + JC)^*(A + JC).$$

(ii) *If n is odd, then the singular values of H are the nonnegative square roots of the eigenvalues of M_3 and the nonnegative square roots of the eigenvalues of M_4, where*

$$M_3 = (A - JC)^*(A - JC),$$

$$M_4 = \begin{bmatrix} |q|^2 + 2x^*x & \sqrt{2}x^*(A + JC) + \sqrt{2}\overline{q}y^T \\ \sqrt{2}\overline{y}q + \sqrt{2}(A + JC)^*x & 2\overline{y}y^T + (A + JC)^*(A + JC) \end{bmatrix}.$$

Theorem 3.11. *Let S be an $n \times n$ skew–centrosymmetric matrix and let S be written as in Lemma 1.3.*

(i) *If n is even, then the singular values of S are the nonnegative square roots of the eigenvalues of M_1 and the nonnegative square roots of the eigenvalues of M_2, where*

$$M_1 = (A - JC)^*(A - JC) \qquad and \qquad M_2 = (A + JC)^*(A + JC).$$

(ii) *If n is odd, then the singular values of S are the nonnegative square roots of the eigenvalues of M_3 and the nonnegative square roots of the eigenvalues of M_4, where*

$$M_3 = (A + JC)^*(A + JC) + 2\overline{y}y^T,$$

$$M_4 = \begin{bmatrix} 2x^*x & \sqrt{2}x^*(A - JC) \\ \sqrt{2}(A - JC)^*x & (A - JC)^*(A - JC) \end{bmatrix}.$$

Proof. Apply Theorem 2.3 of [**1**] to S^*S which is centrosymmetric. □

4. Regular Magic Squares

Let $A = (a_{ij})$ be a regular magic square of order n. Throughout this section, let e be the $n \times 1$ constant vector of ones, $\mu = \frac{n^3 + n}{2}$, and $Z = A - \frac{\mu}{n}ee^T$. Now we discuss the effect of reversing the rows/columns of a regular magic square on its eigenvalues.

Theorem 4.1. *Let A be a regular magic square of order n and let $B = JA$. Then*

(i) *μ is a simple eigenvalue of B and it is the largest eigenvalue of B in magnitude.*

(ii) *$\lambda \neq \mu$ is an eigenvalue of A if and only if $i\lambda$ is an eigenvalue of B.*

(iii) *$(0, x)$ is an eigenpair of A if and only if $(0, x)$ is an eigenpair of B.*

Proof. First, note that $B = JZ + \frac{\mu}{n}ee^T$.

(i) Note that B is also a regular magic square and the trace of B is μ. Thus, μ is a simple eigenvalue of B and it is the largest eigenvalue in magnitude.

(ii) Note that the eigenvalues of A are the same as the eigenvalues of Z, except that the zero eigenvalue corresponding to the eigenvector e, is replaced by μ. Note also that $B = Z' + \frac{\mu}{n}ee^T$, where $Z' = JZ$. Thus, the eigenvalues of B are the same as the eigenvalues of Z', except that the zero eigenvalue corresponding to the eigenvector e, is replaced by μ. But, JZ is skew–centrosymmetric. Thus, by Theorem 3.9, $\text{evals}(Z') = i \cdot \text{evals}(Z)$.

(iii) Note that $B = JA$ and $A = JB$.

$\square$

Thus, if the eigenvalues of A are $\{\mu, \lambda_1, \cdots, \lambda_{n-1}\}$, then the previous theorem implies that the eigenvalues of B are $\{\mu, i\lambda_1, \cdots, i\lambda_{n-1}\}$. Now we use the previous theorem to present our proof of the singularity of regular magic squares of even order.

Corollary 4.2. *Every regular magic square of even order is singular.*

Proof. Let A be a regular magic square of even order n, let $B = JA$, and let the eigenvalues of A be $\{\mu, \lambda_1, \cdots, \lambda_{n-1}\}$. Then, the eigenvalues of B are $\{\mu, i\lambda_1, \cdots, i\lambda_{n-1}\}$. Thus, $\det(A) = \mu \cdot \prod_{j=1}^{n-1} \lambda_j$ and $\det(B) = \pm i \cdot \mu \cdot \prod_{j=1}^{n-1} \lambda_j$. Therefore, $\det(B) = \pm i \det(A)$. But, $\det(B) = \pm \det(A)$. Hence, $\det(A) = 0$. $\square$

Now we prove that $n - 1$ singular values of a regular magic square A of order n are the same as $n - 1$ singular values of Z.

Theorem 4.3. *Let A be a regular magic square of order n and let $B = JA$. Then*

(i) *The singular values of B are the same as the singular values of A.*

(ii) *The singular values of A are the same as the singular values of Z, except that one of the zero singular values of Z is replaced by μ. Moreover, the eigenvectors of $A^T A$ are the same as the eigenvectors of $Z^T Z$.*

Proof.

(i) $B^T B = (JA)^T(JA) = A^T J J A = A^T A$.

(ii) It is easy to prove that (μ^2, e) is an eigenpair of $A^T A$. Here is the proof

$$A^T A e = A^T \mu e = \mu A^T e.$$

Now note that A^T is a regular magic square and note also that the trace of A^T is μ. Thus, (μ, e) is an eigenpair of A^T. Hence, $A^T A e = \mu^2 e$. Now

$$
\begin{aligned}
A^T A &= (Z^T + \frac{\mu}{n}ee^T)(Z + \frac{\mu}{n}ee^T) \\
&= Z^T Z + \frac{\mu}{n}ee^T Z + \frac{\mu}{n}Z^T ee^T + \frac{\mu^2}{n^2}nee^T.
\end{aligned}
$$

Now let (λ, x) be an eigenpair of $A^T A$, where $\lambda \neq \mu^2$. Then $e^T x = 0$, and hence,

$$A^T A x = Z^T Z x + \frac{\mu}{n}ee^T Z x = Z^T Z x.$$

$\square$

References

1. I.T. Abu-Jeib, *Rank–one perturbations and transformations of centrosymmetric matrices*, New Zealand J. Math. **33** (2004), 1–10.
2. I.T. Abu-Jeib, *Centrosymmetric matrices: properties and an alternative approach*, Can. Appl. Math. Q. **10** No. 4 (2002), 429–445.
3. A.L. Andrew, *Eigenvectors of certain matrices*, Linear Algebra and Appl. **7** (1973), 151–162.
4. A.L. Andrew, *Centrosymmetric matrices*, SIAM Rev. **40** (1998), 697–698.
5. A. Cantoni and P. Butler, *Properties of the eigenvectors of persymmetric matrices with applications to communication theory*, IEEE Trans. Comm. **COM-24** No. 8 (1976), 804-809.
6. L. Datta and S.D. Morgera, *On the reducibility of centrosymmetric matrices – applications in engineering problems*, Circuits Systems Signal Process. **8** (1989), 71–96.
7. M.J. Goldstein, *Reduction of the eigenproblem for Hermitian persymmetric matrices*, Math. Comp. **28** (1974), 313–316.
8. I.J. Good, *The inverse of a centrosymmetric matrix*, Technometrics, **12** (1970), 925–928.
9. R.D. Hill, R.G. Bates, and S.R. Waters, *On centro–Hermitian matrices*, SIAM J. Matrix Anal. Appl. **11** No. 1 (1990), 128–133.
10. R.B. Mattingly, *Even order regular magic squares are singular*, Amer. Math. Monthly, **107** No. 9 (2000), 777–782.
11. B. Newman, *MATLAB magic*, 1997 MATLAB Conference, Sydney, Australia (1997), http://www.ceanet.com.au/kb/papers/matlab_magic.htm.
12. F. Stenger, *Matrices of sinc methods*, J. Comput. Appl. Math. **86** (1997), 297–310.
13. W.F. Trench, *Characterization and properties of matrices with generalized symmetry or skew–symmetry*, Linear Algebra Appl. **377** (2004), 207–218.

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